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Erratum

Erratum to: "Modified systems found by symmetry reduction on the cotangent bundle of a loop group" [J. Geom. Phys. 16 (1995) 305–326] ★

Ian Marshall

Department of Mathematics, Leeds University, Leeds LS2 9JT, UK

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Having established that

$$\varphi \circ (\mathcal{J}^L + \mathcal{J}^R, J^L, k, -k\alpha, k((\phi, \phi) - (\theta, \theta))) = \psi \circ J^R \tag{1}$$

it was claimed that φ is necessarily the composition $f \circ \mathbf{m}$ for some $f \in C^\infty(VIR^*)$. Note that this *would* have been correct had J^L and J^R been independent on $T^*\tilde{G}_k$. However they are not independent and their failure to be independent can be expressed by saying that there exist (*nontrivial*) functions of J^L which can be written as functions of J^R, i.e.

$$\exists F, \hat{F} \in C^\infty(\hat{\mathfrak{g}}^*) \quad \text{such that } F \circ J^L = \hat{F} \circ J^R. \tag{2}$$

These functions are precisely the coadjoint invariant functions $I(\hat{\mathfrak{g}}^*)$ on $\hat{\mathfrak{g}}^*$ and indeed one has $F = \hat{F}$.

The correct deduction from formula (1) then is that
either

$$\varphi = f \circ \mathbf{m} \quad \text{for some } f \in C^\infty(VIR^*), \tag{3a}$$

or

$$\varphi(u, \xi, e_1, e - 2, e_3) = F(\xi, e_1) \quad \text{for some } F \in I(\hat{\mathfrak{g}}^*). \tag{3b}$$

In fact it can be very easily proved *without reference* to momentum maps on $T^*\tilde{G}_k$, that the complete set $I(\hat{\mathcal{L}}^*)$ of coadjoint invariants on $\hat{\mathcal{L}}^*$ is generated by functions of the form

$$\varphi = f \circ \mathbf{m} \quad \text{and} \quad \varphi = F \circ \pi \qquad \text{with } f \in I(VIR^*),\ F \in I(\hat{\mathfrak{g}}^*), \tag{4}$$

where π is the projection onto the second factor in the semidirect product, i.e. $\pi(u, \xi, e_1, e_2, e_3) = (\xi, e_1)$. A direct proof follows from the following simple result.

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Lemma 1. *Let $\Sigma = VIR \oplus \hat{\mathfrak{g}}^*$ be the direct sum of the Lie algebras VIR and $\hat{\mathfrak{g}}$, i.e., Σ is based on the vector space*

$$V = (C^\infty(S^1, \mathbb{R}) \oplus \mathbb{R}) \oplus (C^\infty(S^1, \mathfrak{g}) \oplus \mathbb{R}) \tag{5}$$

and the Lie bracket is given by

$$[(\varphi, a_1;\ X, b_1), (\psi, a_2;\ Y, b_2)] = \left(\varphi\psi' - \varphi'\psi, \int \varphi\psi''';\ [XY], \int \operatorname{tr} XY'\right). \tag{6}$$

We may identify the dual Σ^* with V given in (5), *using the pairing*

$$\langle (u, e;\ \xi, \epsilon),\ (\varphi, a;\ X, b)\rangle = \int u\varphi + \int \operatorname{tr} \xi X + ea + \epsilon b. \tag{7}$$

Then the mapping $M : \hat{\mathcal{L}}^* \to \Sigma^*$ given by

$$M(u, \xi, e_1, e_2, e_3) = (\mathbf{m}(u, \xi, e_1, e_2, e_3);\ \xi, e_1) \tag{8}$$

is a Poisson map with respect to the Lie Poisson structures on $\hat{\mathcal{L}}^*$ and on Σ^*. Furthermore, for fixed e_1, e_2, e_3, if $e_1 \neq 0$, M is invertible.

It follows that if $e_1 \neq 0$, the coadjoint invariants on $\hat{\mathcal{L}}^*$ are the pull backs under M^{-1} of the coadjoint invariants on Σ^*.

What are the consequences of the error?

First, the various occurences of the claim: "It is proved that $I(\hat{\mathcal{L}}^*) = \mathbf{m}^* I(VIR^*)$" are simply false.

Second, the introduction of $T^*\tilde{G}_k$, and the natural actions of $\tilde{G}$ and of *Diff* S^1 on it, was unnecessary and indeed irrelevant for finding $I(\hat{\mathcal{L}}^*)$.

Despite these shortcomings the article might be redeemed for the following reasons.

1. The viewpoint one obtains by taking the cotangent bundle $T^*\tilde{G}_k$ as a starting point is still appealing for the nice structures one is given and for the fact that the precise analogy with the finite-dimensional systems such as the Clebsch system follows in a straightforward manner.
2. For the case discussed in Section 5 of [1], which gives the Clebsch system as a special case, there are no new nontrivial invariants. This is because the Lie algebra $\hat{\mathfrak{g}}$ is replaced by the Heisenberg algebra and this has no nontrivial coadjoint invariants, as is easy to check. The results of the paper are therefore totally correct as far as Section 5 goes. (Incidentally, in none of the examples treated in [2] do the "new" invariants on $\hat{\mathcal{L}}^*$ give rise to nontrivial conserved quantities either.)
3. A spin-off of the lemma is that one can very easily define a Lax pair (or zero-curvature representation) for all of the examples discussed in [2] and this is a useful and heretofore unrecognised fact.

Note. Whilst of course on the part of the author the error represents a gross oversight, it needs to be emphasised that it might not have been immediately recognised by a reader that the *principal* motivation for [1] had been the purported proof of the (*false*) claim that

$I(\hat{\mathcal{L}}^*) = \mathbf{m}^*(VIR^*)$. Therefore it was almost certainly very easy for the error to have been missed by the referee .

References

[1] I. Marshall, Modified systems found by symmetry reduction on the cotangent bundle of a loop group, J. Geom. Phys. 16 (1995) 305–326.
[2] I. Marshall, An r-matrix interpretation of modified systems, Physica D 70 (1994) 40–60.